

# THE NON-PARABOLICITY OF INFINITE VOLUME ENDS

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ABSTRACT. Let  $M^m$ , with  $m \geq 3$ , be an  $m$ -dimensional complete non-compact manifold isometrically immersed in a Hadamard manifold  $\bar{M}$ . Assume that the mean curvature vector has finite  $L^p$ -norm, for some  $2 \leq p \leq m$ . We prove that each end of  $M$  must either have finite volume or be non-parabolic.

## 1. INTRODUCTION

Let  $(M^m, \langle \cdot, \cdot \rangle)$  be a complete noncompact Riemannian manifold without boundary. We recall that  $M$  is *parabolic* if it does not admit a non-constant positive superharmonic function. Otherwise, it is said to be *non-parabolic*. There exist equivalent definitions for parabolic manifolds (see for instance Theorem 5.1 of [8]). Let  $E \subset M$  be an *end* of  $M$ , that is an unbounded connected component of  $M - \Omega$ , for some compact subset  $\Omega \subset M$ . The property of parabolicity can be localized on each end of  $M$ . Namely, we say that an end  $E$  is *parabolic* (see Definition 2.4 of [10]) if it does not admit a harmonic function  $f : E \rightarrow \mathbb{R}$  satisfying:

- (1)  $f|_{\partial E} = 1$ ;
- (2)  $\liminf_{y \rightarrow \infty, y \in E} f(y) < 1$ .

Otherwise, we say that  $E$  is a *non-parabolic* end of  $M$ . It is well known that  $M$  is non-parabolic if and only if it admits a non-parabolic end. Furthermore, ends with finite volume are parabolic (see for instance Section 14.4 of [8]). In this direction we recall the following result due to Li and Wang:

**Theorem A** (Corollary 4 of [12] and Corollary 2.9 of [10]). *Let  $E$  be an end of a complete manifold. Suppose that, for some constants  $\nu \geq 1$  and  $C > 0$ ,  $E$  satisfies a Sobolev-type inequality of the form*

$$(1.1) \quad \left( \int_E |u|^{2\nu} \right)^{\frac{1}{\nu}} \leq C \int_E |\nabla u|^2,$$

*for all compactly supported Sobolev function  $u \in W_c^{1,2}(E)$ . Then  $E$  must either have finite volume or be non-parabolic. Moreover, in the case  $\nu > 1$ ,  $E$  must be non-parabolic.*

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Note that if a complete manifold  $M$  that satisfies a Sobolev inequality as in Theorem A with  $\nu = 1$  (that is just the Dirichlet Poincaré inequality) then the first eigenvalue  $\lambda_1(M)$  of the Laplace-Beltrami operator is positive, hence  $M$  must be non-parabolic (see Proposition 10.1 of [8]). Example 4.1 below exhibits a complete manifold that contains a finite volume end and that also satisfies a Sobolev inequality as in Theorem A with  $\nu = 1$ .

Cao, Shen and Zhu [2] showed that if  $M^m$ , with  $m \geq 3$ , is a complete manifold then each end of  $M$  is non-parabolic provided that  $M$  can be realized as a minimal submanifold in a Euclidean space  $\mathbb{R}^n$ . The same conclusion also was obtained by Fu and Xu [7] provided that there exists an isometric immersion of  $M$  in a Hadamard manifold  $\bar{M}$  with finite total mean curvature, that is, the mean curvature vector field  $H$  of the immersion satisfies  $\|H\|_{L^m(M)} < \infty$ . In the both cases, they observed that  $M$  admits a Sobolev-type inequality as in Theorem A with  $\nu > 1$ .

Our main result states the following:

**Theorem 1.1.** *Let  $x : M^m \rightarrow \bar{M}$ , with  $m \geq 3$ , be an isometric immersion of a complete non-compact manifold  $M$  in a Hadamard manifold  $\bar{M}$ . Let  $E$  be an end of  $M$  such that the mean curvature vector satisfies  $\|H\|_{L^p(E)} < \infty$ , for some  $2 \leq p \leq m$ . Then  $E$  must either have finite volume or be non-parabolic.*

Example 4.3 below exhibits an example of a complete non-compact hypersurface  $M^m$  in  $\mathbb{R}^{m+1}$ , with  $m \geq 3$ , of finite volume and mean curvature vector with finite  $L^p$ -norm, for all  $2 \leq p < m - 1$ . This example shows that Theorem 1.1 is not a consequence of Theorem A (except when  $p = m$ ). Note also that the catenoids in  $\mathbb{R}^3$  are parabolic minimal surfaces whose ends have infinite area, which shows that the hypothesis  $m \geq 3$  is essential.

In the present paper we also give a unified proof of the following fact:

**Theorem B.** *Let  $x : M \rightarrow \bar{M}$  be an isometric immersion of a complete non-compact manifold  $M$  in a manifold  $\bar{M}$  with bounded geometry (i.e.,  $\bar{M}$  has sectional curvature bounded from above and injectivity radius bounded from below by a positive constant). Let  $E$  be an end of  $M$  and assume that the mean curvature vector of  $x$  satisfies  $\|H\|_{L^p(E)} < \infty$ , for some  $m \leq p \leq \infty$ . Then  $E$  must have infinite volume.*

The fact above was proved by Frensel [4] and by do Carmo, Wang and Xia [3] for the case that the mean curvature vector field is bounded in norm (the case  $p = \infty$ ), by Fu and Xu [7] for the case that the total mean curvature is finite (the case  $p = m$ ) and by Cheung and Leung [1] for the case that the mean curvature vector has finite  $L^p$ -norm for some  $p > m$ . Since the cylinders of the form  $M^m = \mathbb{S}^{m-1} \times \mathbb{R}$ , where  $\mathbb{S}^{m-1}$  is the unit Euclidean  $(m-1)$ -dimensional sphere, are examples of complete parabolic hypersurfaces in  $\mathbb{R}^{m+1}$  we conclude that boundedness of the mean curvature vector does not imply that  $M$  admits a Sobolev-type inequality. Furthermore, for all  $m \geq 3$ , we exhibit an example of a parabolic complete noncompact hypersurface

$M^m$  in  $\mathbb{R}^{m+1}$  such that the mean curvature vector has finite  $L^p$ -norm, for all  $p > 2(m-1)$ . These examples show that Theorem B is not a consequence of Theorem A.

Two questions arise in this paper: is there an example of a complete noncompact submanifold  $M^m$ , with  $m \geq 3$ , in a Euclidean space satisfying one of the conditions below?

- (1)  $M$  has finite volume and  $\|H\|_{L^p(M)} < \infty$ , for some  $m-1 \leq p < m$ ;
- (2)  $M$  is parabolic and  $\|H\|_{L^p(M)} < \infty$ , for some  $m < p \leq 2(m-1)$ .

## 2. PROOF OF THEOREM 1.1

Choose  $r_0 > 0$  so that the geodesic ball  $B_{r_0} \subset M$  of radius  $r_0$  and center at some point  $\xi_0 \in M$  satisfies  $\partial E \subset B_{r_0}$ . For each  $r > r_0$ , consider  $E_r = E \cap B_r$  and let  $f_r : \overline{E_r} \rightarrow \mathbb{R}$  be a solution of the Dirichlet Problem:

$$\begin{cases} \Delta_M f_r = 0 & \text{in } E_r, \\ f_r = 1 & \text{in } \partial E, \\ f_r = 0 & \text{on } E \cap \partial B_r. \end{cases}$$

It follows from the maximum principle that  $0 < f_r \leq f_s < 1$  in  $E_r$ , for all  $s \geq r$ . Hence, by standard gradient estimates it follows that  $\{f_r\}$  is an equicontinuous family which converges uniformly on compact subsets, when  $r$  goes to infinity, to a function  $f : E \rightarrow \mathbb{R}$  satisfying

$$\begin{cases} \Delta_M f = 0 & \text{in } E, \\ 0 \leq f \leq 1 & \text{in } E, \\ f = 1 & \text{on } \partial E. \end{cases}$$

If  $f \not\equiv 1$  then it follows from the maximum principle that  $\liminf_{x \rightarrow E(\infty)} f(x) < 1$ , which shows that  $E$  is nonparabolic. Furthermore, it is well known that an end of finite volume is parabolic (see section 14.4 of [8]). Hence, to prove Theorem 1.1, it is sufficient to show the following:

*Claim 2.1.* Either  $f \not\equiv 1$  or  $\text{vol}(E)$  is finite.

Suppose, by contradiction, that  $f \equiv 1$  and  $\text{vol}(E)$  is infinite. This implies that, given any  $L > 1$ , there exists  $r_1 > r_0$  such that  $\text{vol}(E_{r_1} - E_{r_0}) > 2L$ . Since  $f_r \rightarrow 1$  uniformly on compact subsets, there exists  $r_2 > r_1$  such that  $f_r^{\frac{2m}{m-2}} > \frac{1}{2}$  everywhere in  $E_{r_1}$ , for all  $r > r_2$ . Thus, defining  $h(r) = \int_{E_r - E_{r_0}} f_r^{\frac{2m}{m-2}}$ , with  $r > r_0$ , we obtain

$$(2.1) \quad h(r) \geq \int_{E_{r_1} - E_{r_0}} f_r^{\frac{2m}{m-2}} > L,$$

for all  $r > r_2$ . In particular, we have that  $\lim_{r \rightarrow \infty} h(r) = \infty$ .

Now, for each  $r > r_0$ , let  $\varphi = \varphi_r \in C_0^\infty(E)$  be a cut-off function satisfying:

- (1)  $0 \leq \varphi \leq 1$  everywhere in  $E$ ;
- (2)  $\varphi \equiv 1$  in  $E_r - E_{r_0}$ .

By Hoffmann-Spruck Inequality [9] we have

$$S^{-1} \left( \int_{E_r} (\varphi f_r)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq \int_{E_r} |\nabla(\varphi f_r)|^2 + \int_{E_r} (\varphi f_r)^2 |H|^2,$$

where  $S$  is a positive constant.

Note that

$$|\nabla(\varphi f_r)|^2 = f_r^2 |\nabla \varphi|^2 + \varphi^2 |\nabla f_r|^2 + \frac{1}{2} \langle \nabla \varphi^2, \nabla f_r^2 \rangle$$

and

$$\varphi^2 |\nabla f_r|^2 = \operatorname{div}_M((f_r \varphi^2) \nabla f_r) - \frac{1}{2} \langle \nabla \varphi^2, \nabla f_r^2 \rangle,$$

since  $f_r$  is harmonic. Using that  $f_r \varphi$  vanishes on  $\partial E_r$  we obtain

$$\begin{aligned} S^{-1} \left( \int_{E_r} (\varphi f_r)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} &\leq \int_{E_r} f_r^2 |\nabla \varphi|^2 + \int_{E_r} \operatorname{div}_M((f_r \varphi^2) \nabla f_r) \\ &\quad + \int_{E_r} (\varphi f_r)^2 |H|^2 \\ &= \int_{E_r} f_r^2 |\nabla \varphi|^2 + \int_{E_r} (\varphi f_r)^2 |H|^2. \end{aligned}$$

Thus, since  $0 \leq \varphi \leq 1$  in  $E$  and  $\varphi \equiv 1$  in  $E_r - E_{r_0}$ , we obtain

$$S^{-1} h(r)^{\frac{m-2}{m}} \leq S^{-1} \left( \int_{E_r} (\varphi f_r)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq \int_{E_{r_0}} f_r^2 |\nabla \varphi|^2 + \int_{E_r} f_r^2 |H|^2. \quad (2.2)$$

First, assume that  $\|H\|_{L^2(E)}$  is finite. Then, since  $0 \leq f_r \leq 1$ , we have

$$S^{-1} h(r)^{\frac{m-2}{m}} \leq \int_{E_{r_0}} |\nabla \varphi|^2 + \int_M |H|^2.$$

Thus,  $\lim_{r \rightarrow \infty} h(r) < \infty$ , which is a contradiction. Now, assume that  $\|H\|_{L^p(E)}$  is finite, for some  $2 < p \leq m$ . Note that  $\frac{m}{m-2} \leq \frac{p}{p-2}$ . Since  $0 \leq f_r \leq 1$  and  $h(r) > 1$ , for all  $r > r_2$ , we have:

- (1)  $f_r^{\frac{2p}{p-2}} \leq f_r^{\frac{2m}{m-2}}$ ;
- (2)  $h(r)^{\frac{p-2}{p}} \leq h(r)^{\frac{m-2}{m}}$ , for all  $r > r_2$ .

Thus, using Hölder Inequality, we have

$$\begin{aligned} (2.3) \quad \int_{E_r - E_{r_0}} f_r^2 |H|^2 &\leq \|H\|_{L^p(E_r - E_{r_0})}^2 \left( \int_{E_r - E_{r_0}} f_r^{\frac{2p}{p-2}} \right)^{\frac{p-2}{p}} \\ &\leq \|H\|_{L^p(E - E_{r_0})}^2 h(r)^{\frac{m-2}{m}}, \end{aligned}$$

for all  $r > r_2$ .

Choose  $r_0 > 0$  large so that  $\|H\|_{L^p(E-E_{r_0})}^2 < \frac{1}{2S}$ . Using (2.2) and (2.3) we obtain the following:

$$S^{-1}h(r)^{\frac{m-2}{m}} \leq \int_{E_{r_0}} |\nabla \varphi|^2 + \int_{E_{r_0}} |H|^2 + \frac{S^{-1}}{2} h(r)^{\frac{m-2}{m}}.$$

This shows that  $\lim_{r \rightarrow \infty} h(r) < \infty$ , which is a contradiction. Therefore, Claim 2.1 and Theorem 1.1 are proved.

### 3. PROOF OF THEOREM B

Since  $\bar{M}$  has bounded geometry, the sectional curvature  $\bar{K}$  and the injectivity radius  $i(\bar{M})$  of  $\bar{M}$  satisfy:

$$(3.1) \quad \bar{K} < b^2 \text{ and } i(\bar{M}) > r_0,$$

for some positive constants  $b$  and  $r_0$ . Let  $E$  be an end of  $M$  and assume that  $\|H\|_{L^p(E)}$  is finite, for some  $m \leq p \leq \infty$ . Fix  $0 < R_0 < \min\{r_0, \frac{\pi}{2b}\}$ , take  $\xi_0 \in M$  and consider  $B_R = B_R(\xi_0)$ , for all  $R > 0$ . Choose  $R_1 > R_0$ , sufficiently large, so that  $\partial E \subset B_{R_1}$  and the distance  $d_M(\partial E, x) > R_0$ , for all  $x$  in  $E - B_{R_1}$ . Let  $q \in \tilde{E} = E - B_{2R_1}$  and  $0 < R < R_0$ . Since  $B_R(q) \subset E - B_{R_1}(\xi_0)$  we obtain, by the isoperimetric inequality for submanifolds (Theorem 2.2 of [9]) and Hölder Inequality, the following:

$$(3.2) \quad S \operatorname{vol}(B_R(q))^{\frac{m-1}{m}} \leq \operatorname{vol}(\partial B_R(q)) + \|H\|_{L^p(E-B_{R_1}(\xi_0))} \operatorname{vol}(B_R(q))^{\frac{p-1}{p}}$$

where  $S > 0$  is a constant that depends only  $m$ .

Assume, by contradiction, that  $\operatorname{vol}(E)$  is finite. Take  $R_1 > 0$  sufficiently large so that

$$(3.3) \quad \operatorname{vol}(E - B_{R_1}) < 1 \text{ and } \|H\|_{L^p(E-B_{R_1})} < \frac{S}{2}.$$

Since  $p \geq m$  we have that  $\frac{p-1}{p} \geq \frac{m-1}{m}$ . By (3.3) and using that  $B_R(q) \subset E - B_{R_1}$  we have that  $\operatorname{vol}(B_R(q))^{\frac{p-1}{p}} \leq \operatorname{vol}(B_R(q))^{\frac{m-1}{m}}$ . Thus, using (3.2) and (3.3), we obtain

$$(3.4) \quad \frac{S}{2} \operatorname{vol}(B_R(q))^{\frac{m-1}{m}} \leq \operatorname{vol}(\partial B_R(q)).$$

By the coarea formula, we have that  $\operatorname{vol}(\partial B_R(q)) = \frac{d}{dR} \operatorname{vol}(B_R(q))$ . Using (3.4), we obtain  $\frac{d}{dR} (\operatorname{vol}(B_R(q)))^{\frac{1}{m}} \geq \frac{S}{2m}$ . This implies that

$$(3.5) \quad \operatorname{vol}(B_R(q)) \geq \frac{S}{2m} R^m,$$

for all  $q \in E_1$  and  $0 < R < R_0$ .

Since  $M$  is complete and  $E \subset M$  is connected and noncompact, there exists a sequence  $p_2, p_3, \dots$  in  $E$  such that

$$(3.6) \quad p_k \in E \cap (B_{2kR_1} - B_{(2k-1)R_1}).$$

Note that  $B_{R_0/2}(p_k) \subset E - B_{R_1}$  and  $B_{R_0/2}(p_k) \cap B_{R_0/2}(p_{k'}) = \emptyset$ , for all  $k \neq k'$ . Since

$$(3.7) \quad \text{vol}(E) \geq \text{vol}(E - B_{R_1}) \geq \sum_{k=2}^{\infty} \text{vol}(B_{R_0/2}(p_k)),$$

it follows from (3.5) that  $\text{vol}(E)$  is infinite. Theorem B is proved.

#### 4. EXAMPLES

**Example 4.1.** Consider the warped product manifold  $M^m = \mathbb{R} \times_{e^t} P$ , where  $P$  is any complete  $(m-1)$ -dimensional manifold with finite volume. The metric of  $M$  is complete and the end  $E = (-\infty, 0) \times P \subset M$  has finite volume given by

$$\text{vol}(E) = \int_{-\infty}^0 \int_P e^{m-1} dt dP = \frac{\text{vol}(P)}{m-1}.$$

Fix  $k \in \mathbb{R}$  and let  $h_\kappa : M \rightarrow \mathbb{R}$  be the function defined by  $h_\kappa(t, x) = \kappa t$ . The gradient vector field of  $h_\kappa$  satisfies

$$(4.1) \quad \nabla h_\kappa = \kappa \frac{\partial}{\partial t},$$

where  $\frac{\partial}{\partial t}(t, x) = \frac{d}{ds} \big|_{s=t} (s, x) \in T_{(t,x)}M$ . It is simple to show that  $\nabla_Z \frac{\partial}{\partial t} = Z - \langle Z, \frac{\partial}{\partial t} \rangle \frac{\partial}{\partial t}$ . This implies that the Laplacian of  $h_\kappa$  satisfies

$$(4.2) \quad \Delta h_\kappa = \kappa \text{div}\left(\frac{\partial}{\partial t}\right) = \kappa(m-1).$$

Fix  $\eta \in C_0^\infty(M)$ . Using (4.1) and (4.2) we obtain

$$\begin{aligned} \kappa(m-1) \int_M \eta^2 &= \int_M \eta^2 \Delta h_\kappa = \int_M (\text{div}(\eta^2 \nabla h_\kappa) - 2\eta \langle \nabla \eta, \nabla h_\kappa \rangle) \\ &= -2 \int_M \langle \nabla \eta, \eta \nabla h_\kappa \rangle \geq - \int_M |\nabla \eta|^2 - \eta^2 |\nabla h_\kappa|^2 \\ &= - \int_M |\nabla \eta|^2 - \kappa^2 \eta^2. \end{aligned}$$

Thus, it holds that  $\int_M |\nabla \eta|^2 + \kappa(\kappa + (m-1))\eta^2 \geq 0$ , for all  $k \in \mathbb{R}$ . In particular, if we take  $\kappa = -\frac{m-1}{2}$  we obtain

$$\int_M |\nabla \eta|^2 - \frac{(m-1)^2}{4} \eta^2 \geq 0.$$

Hence  $M$  satisfies a Sobolev inequality as in Theorem A with  $\nu = 1$ .

**Example 4.2.** Let  $f : (-\infty, \infty) \rightarrow (0, \infty)$  be a positive smooth function satisfying that  $f(t) = f(-t)$  and  $f(t) = t^{\frac{1}{m-1}}$ , for all  $t \geq 1$ . Consider the immersion  $x : \mathbb{S}^{m-1} \times \mathbb{R} \rightarrow \mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R}$  given by  $x(v, t) = (f(t)v, t)$ .

Consider  $M$  the product  $\mathbb{S}^{m-1} \times \mathbb{R}$  endowed with the metric induced by  $x$ . The metric of  $M$  is given by

$$(4.3) \quad \langle \cdot, \cdot \rangle_{(v,t)} = (1 + f'(t)^2) dt^2 + f(t)^2 \langle \cdot, \cdot \rangle_v,$$

where  $\langle \cdot, \cdot \rangle_v$  denotes the metric of  $\mathbb{S}^{m-1}$ . Note that  $M$  is a complete manifold with two ends.

We claim that  $M$  is parabolic. To do this, it is sufficient to prove that the following ends of  $M$ :

$$E_+ = (1, \infty) \times \mathbb{S}^{m-1} \text{ and } E_- = (-\infty, -1) \times \mathbb{S}^{m-1}.$$

are parabolic (see Proposition 14.1 of [8]). In fact, we define:

$$V_+(s) = \text{vol}_M(\{q \in E_+ \mid d(q, \partial E_+) \leq s\})$$

and

$$V_-(s) = \text{vol}_M(\{q \in E_- \mid d(q, \partial E_-) \leq s\})$$

Using (4.3) and that  $f(t) = t^{\frac{1}{m-1}}$ , for all  $|t| \geq 1$ , we obtain that

$$V_+(s) = V_-(s) \leq Ds^2,$$

for some constant  $D > 0$  and for all  $s \geq 1$ . In particular,

$$\int_1^\infty \frac{s}{V_+(s)} ds = \int_1^\infty \frac{s}{V_-(s)} ds = \infty.$$

This implies that  $M$  is parabolic (see section 14.4 of [8]).

We claim that the mean curvature vector  $H$  of the isometric immersion  $x$  has finite  $L^p$ -norm, for all  $p > m$ . In fact, a simple computation shows that

$$(4.4) \quad mH(x(v, t)) = \frac{(m-1)}{f(t)\sqrt{1+f'(t)^2}} - \frac{f''(t)}{(1+f'(t)^2)^{\frac{3}{2}}}$$

Using that  $f(t) = t^{\frac{1}{m-1}}$ , for all  $|t| \geq 1$ , we obtain that  $|H(x(v, t))| \leq Ct^{-\frac{1}{m-1}}$ , for some  $C > 0$  and for all  $x(v, t) \in E_+ \cup E_-$ . Thus, we obtain

$$\int_M |H|^p dM \leq D \int_1^\infty t^{1-\frac{p}{m-1}} dt,$$

for some  $D > 0$ . This implies that  $\|H\|_{L^p(M)}$  is finite when  $p > 2(m-1)$ .

**Example 4.3.** Let  $x : \mathbb{S}^{m-1} \times \mathbb{R} \rightarrow \mathbb{R}^{m+1}$  be the immersion given by  $x(v, t) = (e^{-t^2}v, t)$  and consider  $M = \mathbb{S}^{m-1} \times \mathbb{R}$  endowed with the metric induced by  $x$ . The metric of  $M$  is complete and the volume of  $M$  is given by

$$(4.5) \quad \text{vol}(M) = \omega_{m-1} \int_{-\infty}^{+\infty} (1 + 4t^2 e^{-2t^2})^{\frac{1}{2}} e^{-(m-1)t^2} dt,$$

where  $\omega_{m-1}$  is the volume of  $\mathbb{S}^{m-1}$ . This implies that  $\text{vol}(M)$  is finite, since the integral  $\int_{-\infty}^{+\infty} e^{-(m-1)t^2} dt$  is finite and the function  $t \in \mathbb{R} \mapsto 1 + 4t^2 e^{-2t^2}$  is bounded. In particular,  $M$  is parabolic since it has finite volume (see Theorem 7.3 of [8]).

The mean curvature vector  $H$  of the isometric immersion  $x$  is given by

$$(4.6) \quad H(x(v, t)) = h(t) = \frac{2e^{-t^2}(1 - 2t^2)}{m(4t^2e^{-2t^2} + 1)^{\frac{3}{2}}} + \frac{(m - 1)e^{t^2}}{m(4t^2e^{-2t^2} + 1)^{\frac{1}{2}}}.$$

Using that  $\lim_{t \rightarrow \infty} e^{-t^2}(1 - 2t^2) = \lim_{t \rightarrow \infty} 4t^2e^{-2t^2} = 0$  we obtain that

$$(4.7) \quad \lim_{t \rightarrow \pm\infty} h(t)e^{-t^2} = \frac{m - 1}{m}.$$

Thus the integral

$$(4.8) \quad \begin{aligned} \int_M |H|^p &= \omega_{n-1} \int_{-\infty}^{+\infty} \left( |h(t)|^p (1 + 4t^2e^{-2t^2})^{\frac{1}{2}} e^{-(m-1)t^2} \right) dt \\ &= \omega_{n-1} \int_{-\infty}^{+\infty} \left( (|h(t)|e^{-t^2})^p (1 + 4t^2e^{-2t^2})^{\frac{1}{2}} \right) e^{(p-m+1)t^2} dt \end{aligned}$$

converges if  $0 \leq p < m - 1$  and diverges if  $p \geq m - 1$ .

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